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THERMAL SLIP FOR A GAS WITH COLLISION FREQUENCY PROPORTIONAL TO THE VELOCITY OF MOLECULES

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Exact solutions are obtained for the problem of the thermal slip of an inhomogeneously heated gas along a plane surface in a half-space. Two classes of the model kinetic Boltzmann equations are applied: the equation with a collision operator in the BGK (Bhatnagar, Gross, and Krook) form and in the form of an ellipsoidal-statistical model (this equation is constructed for the first time). The collision frequency in both models is proportional to the velocity of molecules. The results of numerical calculations are presented.

Kinetic equations are generally applied with a constant frequency of collisions. However, the hypothesis that the frequency of the collisions of molecules is independent of their velocity is a substantial simplification. For real gases, the frequency of collisions is a rather complicated function of the velocity of molecules. The assumption on the constancy of the free-path length of molecules (at least for those whose interaction can be approximated by a model of solid spheres) seems to be more realistic. This assumption is equivalent to the hypothesis that the frequency of collisions of molecules is proportional to their velocity.

In the present work, to solve the problem of determining the thermal slip velocity, for the first time we construct the Boltzmann linearized equation with a collision operator in the form of an ellipsoidal-statistical model with frequency proportional to the velocity of molecules (the ES-equation). An exact solution of this problem is obtained. As a particular case, from this we derive the solution of this problem for the BGK-equation of Boltzmann. It is precisely this model that was already used for solving the Smolukhovskii problem of a temperature jump and other problems [1-5]. However, the BGK-model possesses a well-known drawback, i.e., it leads to the wrong Prandtl number: Pr = 1. The ES-equation, which is constructed here for the first time, gives the true Prandtl number: Pr = 2/3. More precisely, in this case we construct a set of equations with a parameter that is linearly related to the Prandtl number, and develop a method of constructing exact solutions for this set of equations.

We recall that the thermal slip of a gas is the motion of a gas along a surface that is caused by a temperature gradient [6]. The study of thermal slip applying equations with a constant frequency of collisions is the concern of an extensive literature (see [1, 7] and references therein). Thus, in [7], the thermal-slip velocity was determined on the basis of the BGK-equation with a constant frequency of collisions, while in [8] this was done on the basis of the Boltzmann model equation with a collision operator of mixed type, suggested by Shakhov in [9].

Suppose in the half-space $x > 0$ filled with a monoatomic gas there is a linear temperature gradient far from the wall. Let us introduce a Cartesian coordinate system with the center on the surface, with the x axis perpendicular to the surface, and with the y axis directed along the gas motion. We are to find the slip velocity of the gas caused by the temperature gradient.

First, we construct the ES-equation with a constant collision frequency for the given problem. For a stationary case, the BGK-equation can be written in the form

$$v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} = \nu \left(f_{\text{eq}} - f\right) , \tag{1}$$

where

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$$f_{\mathrm{eq}} = n\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left\{-\frac{m}{2kT}\,[v_x^2 + (v_y - u^*)^2 + v_z^2]\right\}.$$

We limit ourselves to a slow (compared to the speed of sound) motion of the gas and to small gradients. In this case, the problem admits linearization:

$$f = f_0\,(1 + \varphi)\,,$$

where

$$f_{\mathrm{eq}} = f_0\,[1 + K_T y\,(c^2 - 5/2) + 2c_y u\,(x)]\,,$$

$$f_0 = n_0\left(\frac{m}{2\pi kT_0}\right)^{3/2} \exp\,(-c^2)\,,\quad K_T = \left(\frac{\partial \ln T}{\partial y}\right)_{x=\infty}\,,$$

$$\mathrm{c} = \beta_0^{1/2}\mathrm{v}\,,\quad u\,(x) = \beta_0^{1/2}\,u^*\,(x)\,,\quad \beta_0 = m/(2kT_0)\,.$$

Here, for the function φ Eq. (1) has the form

$$c_x\frac{\partial\varphi}{\partial x} + c_y\frac{\partial\varphi}{\partial y} = \nu\left[K_T y\left(c^2 - \frac{5}{2}\right) + 2c_y\,u\,(x) - \varphi\right]. \tag{2}$$

Equation (2) should satisfy conservation laws. The only nontrivial conservation law in the given problem is the law of conservation of momentum. From this law it follows that

$$\int \nu\,[2c_y\,u\,(x) - \varphi\,]\,c_y \exp\,(-c^2)\,d^3c = 0 \tag{3}$$

(the integration extends over the entire space of velocities). When $\nu = \mathrm{const}$, Eq. (3) yields

$$u\,(x) = \pi^{-3/2}\int \varphi\,(x, y, \mathrm{c})\,c_y \exp\,(-c^2)\,d^3c\,, \tag{4}$$

which corresponds to the usual determination of the mass velocity of the gas.

The right-hand side of Eq. (2) can be considered as the first terms in the expansion of the linearized collision integral into polynomials of velocity. Considering the next moment of the distribution function and assuming that

$$\varphi = h\,(x, \mathrm{c}) + K_T\left(y - \frac{1}{\nu}\,c_y\right)\left(c^2 - \frac{5}{2}\right) - \frac{2}{3\sqrt{\pi}\,\nu}\,K_T\,c_y\,,$$

we obtain the equation

$$c_x\frac{\partial h}{\partial x} = \nu\,[2c_y u\,(x) + bc_xc_yP_{xy}\,(x) - h\,]\,, \tag{5}$$

where

$$P_{xy}\,(x) = \pi^{-3/2}\int h\,(x, \mathrm{c})\,c_xc_y \exp\,(-c^2)\,d^3c\,,$$

and the number b is the parameter connected with the Prandtl number by the relation $\mathrm{Pr} = 8/9 + 5/6\sqrt{\pi}\,b$.

We will consider now the case where the frequency is proportional to the absolute value of the velocity of molecules, i.e., $\nu = \nu_0 c$. In this case, the general form (Eq. (5)) of the BGK-equation is retained, but, according to Eqs. (3) and (4), now we have

$$u(x) = \frac{3}{8\pi} \int h(x, \mathrm{c})\, c_y c \exp(-c^2)\, d^3c .$$

Correspondingly, the ES-equation can be written in the form

$$c_x \frac{\partial h}{\partial x} = \nu_0 c\, [2c_y u(x) + bc_x c_y P_{xy}(x) - h] ,$$

where

$$P_{xy}(x) = \int h(x, \mathrm{c})\, c_x c_y c \exp(-c^2)\, d^3c .$$

We will assume that the molecules are reflected from the wall in a purely diffuse manner; then

$$h(0, \mathrm{c}) = \frac{K_T c_y}{\nu_0 c} \left(c^2 - \frac{5}{2} \right) - 2u_0 c_y + \frac{2K_T c_y}{3\sqrt{\pi}\,\nu_0} , \quad c_x > 0 ;$$

moreover, in the given problem $h(\infty, \mathrm{c}) = 0$.

Representing the function h as $h = c_y \psi(x, \mathrm{c})$, we obtain the equation

$$c_x \frac{\partial \psi}{\partial x} = \nu_0 c\, [2u(x) + bc_x P_{xy}(x) - \psi] ,$$

in which

$$u(x) = \frac{3}{8\pi} \int \psi(x, \mathrm{c})\, c_y^2 \exp(-c^2)\, d^3c ,$$

$$P_{xy}(x) = \int \psi(x, \mathrm{c})\, c_x c_y^2 c \exp(-c^2)\, d^3c .$$

We now pass to a spherical coordinate system in the space of velocities, introducing a polar axis along c_x and assuming that $\mu = \cos\theta = c_x/c$. In the last two integrals, we perform integration over a polar angle and replacing again $\nu_0 x$ by x and K_T/ν_0 by K_T, we obtain the equation

$$\mu \frac{\partial}{\partial x} \psi(x, \mu, c) + \psi(x, \mu, c) =$$

$$= \frac{3}{4} \int_0^\infty c'^5 \exp(-c'^2)\, dc' \int_{-1}^{1} (1 - \mu'^2)(1 + \beta\mu c \mu' c')\, \psi(x, \mu', c')\, d\mu' , \tag{6}$$

in which

$$\beta = \frac{8}{3}\pi b , \quad \mathrm{Pr} = \frac{8}{9} + \frac{25\pi\beta}{32(5 - 3\beta)} .$$

If we seek the solution of Eq. (6) in the form

$$\psi = \psi_1(x, \mu) + c\psi_2(x, \mu) + \left(\frac{1}{c} + N_0 + N_1 c \right) \psi_3(x, \mu) ,$$

then we obtain the following set of three equations.

$$\mu \frac{\partial \psi_1}{\partial x} + \psi_1(x, \mu) = \frac{3}{4} \int_{-1}^{1} (1 - \mu'^2)\, [\psi_1(x, \mu') + 5\alpha\psi_2(x, \mu') +$$

$$+ (N_0 + 5\alpha N_1 + 2\alpha)\, \psi_3 (x, \mu')\,]\, d\mu' ,$$

$$\mu \frac{\partial \psi_2}{\partial x} + \psi_2 (x, \mu) = \frac{3}{4} 5\alpha\beta\mu \int_{-1}^{1} \mu' (1 - \mu'^2) [\psi_1 (x, \mu') + 3\psi_2 (x, \mu') +$$

$$+ (5\alpha N_0 + 3N_1 + 1)\, \psi_3 (x, \mu')\,]\, d\mu' ,$$

$$\mu \frac{\partial \psi_3}{\partial x} + \psi_3 (x, \mu') = 0 , \quad \alpha = \frac{3}{16} \sqrt{\pi} .$$

We will select the numbers N_0 and N_1 so that the following equations are satisfied:

$$N_0 + 5\alpha N_1 + 2\alpha = 0 \quad \text{and} \quad 5\alpha N_0 + 3N_1 + 1 = 0 .$$

Then, the function $\psi_3(x, \mu)$ drops out of the first two equations, whereas the third equation does not make a contribution to the slip velocity. We will write the first two equations in vector form:

$$\mu \frac{\partial}{\partial x} \Psi (x, \mu) + \Psi (x, \mu) = \frac{3}{4} \int_{-1}^{1} (1 - \mu'^2)\, K (\mu, \mu')\, \Psi (x, \mu')\, d\mu' , \tag{7}$$

where

$$\Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} , \quad K (\mu, \mu') = K_0 + \beta\mu\mu' K_1 ,$$

$$K_0 = \begin{bmatrix} 1 & \gamma \\ 0 & 0 \end{bmatrix} , \quad K_1 = \begin{bmatrix} 0 & 0 \\ \gamma & 3 \end{bmatrix} , \quad \gamma = 5\alpha .$$

The boundary conditions for Eq. (7), formulated previously for h, have the form

$$\Psi (0, \mu) = \Psi_0 , \quad 0 < \mu < 1 ; \tag{8}$$

$$\Psi (\infty, \mu) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} , \quad -1 < \mu < 0 , \tag{9}$$

where

$$\Psi_0 = \begin{bmatrix} -2u_0 + \left(\dfrac{2}{3\sqrt{\pi}} + \dfrac{5}{2} N_0 \right) K_T \\ \left(1 + \dfrac{5}{2} N_1 \right) K_T \end{bmatrix} .$$

Further, we will assume that $K_T = 1$, so that u_0 is the desired coefficient of the thermal slip velocity. Separating the variables

$$\Psi_\eta (x, \mu) = \exp \left(- \frac{x}{\eta} \right) \Phi (\eta, \mu)$$

after certain transformations reduces Eq. (7) to the characteristic equation:

$$(\eta - \mu)\, \Phi (\eta, \mu) = \frac{3}{4} \eta \Delta (\mu\eta)\, n (\eta) , \tag{10}$$

where

$$n(\eta) = \int_{-1}^{1} (1-\mu^2)\, \Phi(\eta,\mu)\, d\mu\,, \tag{11}$$

$$\Delta(x) = \begin{bmatrix} 1 & \gamma \\ 0 & c\beta x \end{bmatrix}, \quad c = 3 - \gamma^2 .$$

Let $\eta \in (-1; 1)$. Then from Eqs. (10) and (11) in the space of generalized functions (see [10]) we find the eigenvectors of the continuous spectrum

$$\Phi(\eta,\mu) = F(\eta,\mu)\, n(\eta)\,,$$

where

$$F(\eta,\mu) = \frac{3}{4}\eta\Delta(\mu\eta)\, P\frac{1}{\eta-\mu} + (1-\eta^2)^{-1}\, \Lambda(\eta)\, \delta(\eta-\mu)\,;$$

$$\Lambda(z) = I + \frac{3}{4} z \int_{-1}^{1} (1-\mu^2)\, \Delta(\mu z)\, \frac{d\mu}{\mu - z}$$

is the dispersion matrix; the symbol Px^{-1} denotes the distribution, i.e., the principal value of the integral of x^{-1}; $\delta(x)$ is the delta-function.

We find the dispersion function of the problem $\lambda(z) = \det \Lambda(z)$:

$$\lambda(z) = \Omega_1(z)\, \Omega_2(z)\,,$$

where

$$\Omega_1(z) = -\frac{1}{2} + \frac{3}{2}(1-z^2)\lambda_C(z)\,, \quad \Omega_2(z) = 1 + c\beta z^2\, \Omega_1(z)\,; \quad \lambda_C(z) = \frac{1}{2}\int_{-1}^{1} \frac{\tau d\tau}{\tau - z}\,,$$

$\lambda_C(z)$ is the Keyes dispersion function.

We note that the function $\lambda(z)$ has the double zero at the point $z = \infty$. Two characteristic solutions of Eq. (7) correspond to this zero:

$$\Psi^{(1)}(x,\mu) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \Psi^{(2)}(x,\mu) = \begin{bmatrix} x-\mu \\ -b\mu \end{bmatrix}, \quad b = \frac{\gamma\beta}{5-3\beta}\,.$$

By applying the principle of argument (see [11]), we can show that $\lambda(z)$ has no other zeros.

We will show that the boundary-value problem (7)-(9) has a solution representable in the form of expansion in the eigenvectors of characteristic Eq. (10):

$$\Psi(x,\mu) = \int_{0}^{1} \exp\left(-\frac{x}{\eta}\right) F(\eta,\mu)\, a(\eta)\, d\eta\,, \tag{12}$$

where $a(\eta)$ is the unknown vector-function with the elements $a_1(\eta)$ and $a_2(\eta)$.

Using the boundary conditions, we will go over from expansion (12) to the singular integral equation with the Cauchy kernel:

$$-\Psi_0 + \frac{3}{4}\int_{0}^{1} \eta\Delta(\mu\eta)\ a(\eta)\, \frac{d\eta}{\eta-\mu} + (1-\mu^2)^{-1}\, \Lambda(\mu)\, a(\mu) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad 0 < \mu < 1\,. \tag{13}$$

We will introduce the auxiliary vector-function

$$N(z) = \begin{bmatrix} N_1(z) \\ N_2(z) \end{bmatrix} = \int_0^1 \eta\Delta(\eta z)\, a(\eta) \frac{d\eta}{\eta - z}. \tag{14}$$

Using Sokhotskii's formulas [11] for $N(z)$ and $B(z) = \Lambda(z)\Delta^{-1}(z^2)$, we transform Eq. (13) to the Riemann–Hilbert vector boundary-value problem:

$$B^+(\mu)\,[N^+(\mu) - \Psi_0] = B^-(\mu)\,[N^-(\mu) - \Psi_0], \quad 0 < \mu < 1. \tag{15}$$

For this, we will first solve the corresponding factorization problem

$$B^+(\mu)\,X^+(\mu) = B^-(\mu)\,X^-(\mu), \quad 0 < \mu < 1. \tag{16}$$

The solution of problem (16) will be sought in the form of $X(z) = SU(z)S^{-1}$, where $U = \operatorname{diag}\{U_1, U_2\}$ is the new unknown matrix, whereas the matrix S reduces the matrix $B(z)$ to a diagonal form, i.e.,

$$S^{-1} B(z)\, S = \Omega(z) = \operatorname{diag}\left\{\Omega_1(z),\, \Omega_2(z)\right\}.$$

Substituting $X(z)$ into Eq. (16), we obtain the matrix boundary-value problem, which is equivalent to two scalar problems:

$$\Omega_\alpha^+(\mu)\, U_\alpha^+(\mu) = \Omega_\alpha^-(\mu)\, U_\alpha^-(\mu), \quad \alpha = 1, 2, \quad 0 < \mu < 1.$$

Investigating the properties of the functions $\Omega_\alpha^+(\mu)$, $\alpha = 1, 2$, we obtain the solutions, limited at the end points of the integration interval:

$$U_\alpha(z) = z^{\kappa_\alpha} \exp[-V_\alpha(z)],$$

where

$$V_\alpha(z) = \frac{1}{\pi} \int_0^1 [\theta_\alpha(\mu) - \kappa_\alpha \pi] \frac{d\tau}{\tau - z}, \quad \kappa_1 = 1, \quad \kappa_2 = 0;$$

$$\theta_\alpha(\tau) = \arg \Omega_\alpha^+(\tau)$$

are the principal values of the arguments.

Thus, the matrix $X(z)$ is constructed and has the form

$$X(z) = \begin{bmatrix} U_1(z) & \gamma\,[U_1(z) - U_2(z)] \\ 0 & U_2(z) \end{bmatrix}.$$

Now, we transform problem (15) by using factorization (16) and obtain

$$[X^+(\mu)]^{-1}\,[N^+(\mu) - \Psi_0] = [X^-(\mu)]^{-1}\,[N^-(\mu) - \Psi_0], \quad 0 < \mu < 1. \tag{17}$$

Taking into account the behavior of the matrices and vectors entering into Eq. (17), we find a general solution of this problem:

$$N(z) = \Psi_0 + X(z) \begin{bmatrix} c_0 \\ d_0 \end{bmatrix}, \tag{18}$$

where c_0 and d_0 are arbitrary constants.

In order that solution (18) could be taken as the auxiliary function $N(z)$, defined by equality (14), we will eliminate the pole in the function $N_1(z)$ at the point $z = \infty$ and make it vanishing at this point. Then we equate the limits of the function $N_2(z)$ at the point $z = \infty$. After this we obtain

$$c_0 = -\gamma d_0,$$

$$u_0 = \frac{1}{3\sqrt{\pi}} + \frac{5}{4} N_0 - \frac{1}{2}\gamma d_0, \tag{19}$$

$$-c\beta \int_0^1 \eta^2 a_2(\eta)\, d\eta = 1 + \frac{5}{2} N_1 + d_0. \tag{20}$$

By applying the Sokhotskii formula [11] to the function $N(z)$, defined by equality (18), we find the coefficient of the continuous spectrum $a(\eta)$:

$$2\pi i \eta a(\eta) = \Delta^{-1}(\eta^2)\,[X^+(\eta) - X^-(\eta)] \begin{bmatrix} c_0 \\ d_0 \end{bmatrix}.$$

Now, equality (20) yields the coefficient

$$d_0 = -\left(1 + \frac{5}{2} N_1\right)(1 + I_0)^{-1},$$

where

$$I_0 = \frac{1}{2\pi i} \int_0^1 [U^+(\eta) - U^-(\eta)] \frac{d\eta}{\eta}.$$

Using a technique of contour integration, we obtain that $I_0 = U_2(0) - 1$. To calculate $U_2(0)$, we make use of the factorization

$$\Omega_2(z) = c\beta\, \Omega_2(\infty)\, z^2\, U_2(z)\, U_2(-z).$$

This formula is derived either directly or by the same method that was used in [12, p. 149]. From the latter equality we find that

$$U_2(0) = \left(1 - \frac{1}{5} c\beta\right)^{-1/2}.$$

Now, all the free parameters of solution (18) are determined, including the coefficient of the thermal slip velocity (see formula (19)):

$$u_0 = \frac{1}{3\sqrt{\pi}} - \frac{\gamma}{4(3 - \gamma^2)} \frac{I_0}{1 + I_0}. \tag{21}$$

The validity of expansion (12) is established, since the coefficient of the continuous spectrum $a(\eta)$ is found uniquely.

We note that when $\beta \to 0$, Eq. (6) and, consequently, Eq. (7) go over into the corresponding BGK-equation (see, for example, [1, 5]). From the expression for $U_2(0)$, it is evident that $U_2(0) = 1$ at $\beta = 0$, consequently, $I_0 = 0$; therefore, from formula (21) we have

$$u_0 = \frac{1}{3\sqrt{\pi}}. \tag{22}$$

It remains to go over to dimensional quantities. The true velocity of thermal slip is calculated from the formula

$$v_{sl} = \nu K_{Tsl} \left(\frac{\partial \ln T}{\partial y}\right)_{x=\infty},$$

where ν is the kinematic viscosity, $K_{Tsl} = (5\sqrt{\pi}/3\mathrm{Pr})u_0$, with the value of u_0 being determined according to Eq. (21) for the ES-model and according to Eq. (22) for the BGK-model. Numerical calculations show that $K_{Tsl} = 0.94677$ for the ES-model and $K_{Tsl} = 0.83333$ for the BGK-model.

We will note for comparison that the exact value of the thermal-slip coefficient, obtained in [13] by numerical methods on the basis of the total Boltzmann equation, is equal to $K_{Tsl}^{(0)} = 1.00867$. For the BGK-model with constant collision frequency [7], this coefficient is equal to $K_{Tsl} = 1.149$, which is higher by 14% than $K_{Tsl}^{(0)}$. The ES-model with a constant frequency gives the same result as the BGK-model. It follows from the aforegoing that the result obtained for K_{Tsl} by using the BGK-model with the collision frequency proportional to the absolute value of velocity differs from $K_{Tsl}^{(0)}$ by 17.5%, whereas that obtained by using the corresponding ES-model differs only by 6.1%. Thus, account for the dependence of the collision frequency on velocity in the ES-model makes it possible to increase the accuracy of the description of thermal slip by more than two times.

In conclusion, it should be noted that in addition to solving a specific physical problem in the present work, we continue to develop a method for solving boundary-value problems of the kinetic theory in the form of expansion into generalized singular eigenvectors of a corresponding characteristic equation. The expansion of the solution sought is reduced to solving the Riemann–Hilbert vector boundary-value problem. The corresponding homogeneous problem is considered (factorization of its coefficient). By diagonalizing the coefficient, the problem is reduced to two scalar boundary-value problems, whose solution has already been found by standard methods [11]. The solvability conditions allow one to determine all the free solution parameters uniquely, whereas from the Sokhotskii formulas for the difference of the boundary values of solution, one can uniquely derive the coefficient of a continuous spectrum. Thus, the validity of the expansion of the solution into eigenvectors is established.

In recent years, the method described made it possible to obtain a series of significant results in the kinetic theory (see, for example, [14-18]). The method developed can be used for solving various problems arising in interaction of a gas and plasma with a surface.

NOTATION

f, distribution function; m, mass of a molecule; T, temperature; n, concentration; v, velocity of molecules; $u(x)$, mass velocity of the gas; k, Boltzmann constant; ν, frequency of collisions; f_{eq}, equilibrium distribution function; K_T, logarithmic gradient of temperature; K_{Tsl}, coefficient of thermal slip velocity.

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